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REVIEW

Theoretical Glaciology. By K. Hutter. Reidel, 1983. 510 pp. Dfl. 240.

Glaciology is the study of the large-scale behaviour of natural ice masses, ranging from valley glaciers to grounded polar ice sheets and floating ice shelves. It is a multidisciplinary science, with major empirical contributions from geography and geomorphology standing alongside experimental and theoretical geophysics. Observations of slow, gravity-driven, large-scale ice flow, and of the effects of past ice movements over longer timescales, have provided the stimulus to develop flow models that describe the observed phenomena, and which can be used to predict results for alternative flow configurations. They have an important role in the understanding of past climate. Such modelling requires a constitutive theory for the thermomechanical response of ice to stress and heating in conjunction with the mass, momentum and energy balances of continuum physics, together with a proper formulation of thermomechanical boundary conditions at a grounded sheet base, at a submerged floating shelf base, on a free surface subjected to accumulation or ablation, and at a phase-change interface between cold and temperate ice zones. The resulting boundary-value problems for even the most simple idealized flow configurations call for non-trivial analytical and numerical techniques. Mathematics has played an increasing role in recent years, and must remain a key ingredient in the future development of glaciology. This is the theme of Dr Hutter's text.

The book provides a self-contained account of theoretical glaciology which reflects developments up to 1982, and includes applications and results not published elsewhere. A full treatment of the underlying continuum thermodynamics is incorporated. While the emphasis is on the mathematical rigour of both the formulation of theoretical mechanics models and the methods of solution, the physical background and motivation is presented at every stage. There are extensive references to earlier models based on more arbitrary assumptions, which have been the standards for glaciological applications, and an important feature is their comparison with more recent systematic approximation schemes to demonstrate both merits and limitations. Detailed analyses are presented for various model flow configurations for which, in appropriately normalized dimensionless variables, a small parameter arises through an imposed perturbation on an elementary flow solution, or through coordinate stretchings introduced to reflect the status of different spatial and time derivatives in the balances. Series expansions in the parameter then display the leading-order balance equations (or reduced model) against which alternative approximations can be judged, and yield equations for the first order corrections if required. The wide applications of small-surface-slope (relative to mean bed inclination) models are important practical examples. Numerous illustrations and conclusions are provided to demonstrate the values of rational approximation methods.

Chapter 1 presents the mathematical formulation of the kinematics, kinetics and thermodynamics of a continuum description of material behaviour, together with the invariance principles for constitutive relations defining classes of material properties. Attention is given to jump conditions across a singular surface, anticipating non-material boundaries with accumulation or ablation and internal phase-change interfaces, and details are presented for heat-conducting compressible and incompressible viscous fluids. The notation and derivation of the continuum results required in the main text are therefore available without external search, which should be helpful to the many glaciology practitioners not familiar with this material. The discussion of constitutive relations for ice in Chapter 2 covers the instantaneous elasticity and the short-time primary, secondary and tertiary viscoelastic creep, in addition to the long-time viscous response. This treatment provides the essential background for the ice mechanics of cold-regions engineering, which is of increasing practical importance. Laboratory data, necessarily restricted to the short-time response (relative to the timescales of years to thousands of years for glacier response) and to small samples, show wide discrepancies between various data sets. Furthermore, the tests are mainly confined to uniaxial compressive stress, so that the tensor relation describing response to arbitrary stress requires further (unconfirmed) assumptions. The conventional model of an incompressible, nonlinearly viscous, heat-conducting fluid for the long-time glacier response rests on an interpretation of laboratory data together with ice-core measurements. It includes a strongly temperature-dependent rate factor and the assumption of parallel deviatoric stress and strain rate. Strain-rate measurements in ice shelves have already suggested that the large-scale response of natural ice is significantly more viscous than the models deduced from laboratory tests, so the analysis of natural ice flows with associated data correlation is necessary to establish a more satisfactory constitutive model, including the significant temperature dependence.

In Chapter 3 the balance equations and boundary conditions for both cold (below-melting) and temperate (at-melting) ice regions are presented. While the conventional viscous model is adopted for cold ice, a new feature in glaciology is the description of temperate ice by a mixture theory with ice and water phases. It is suggested that a diffusion type law for moisture flux is necessary (and needs to be established) for a satisfactory physical description. Moisture content replaces temperature (now given by the melting condition) as an independent variable in the ice law, but its influence on the mixture properties is not fully established. It is shown from the jump conditions that traction is not continuous at a free surface subject to accumulation or ablation, though the discontinuity is negligible in practice, and that the heat-flux discontinuities at freezing and melting surfaces are different. There is a discussion of ice–bedrock conditions, specifically the tangential condition and the difference between cold ice frozen to the bed and temperate ice sliding over a thin water layer, while thermal conditions at the base of a floating shelf are described with more than customary care. The chapter continues by illustrating the cold ice and the temperate ice equations for steady, plane, gravity-driven flow in a parallel-sided slab. Dimensionless variables are introduced, giving rise to a variety of dimensionless parameters whose magnitude ranges are estimated for practical conditions so that the significance of different terms in the balances are illuminated. In particular, inertia terms are always negligible. Some simple solutions are obtained for stress, velocity and temperature independent of the longitudinal coordinate, and are discussed in terms of power-law creep. The inclusion of longitudinal strain rate to describe extending and compressing flows reveals the inconsistency of the earlier treatment.

Chapter 4 presents a perturbation analysis to describe small variations of thickness over a lengthscale greater than the thickness but much smaller than the span, with the parallel-sided slab solution as the zeroth-order approximation. That is, both the bedrock profile and surface accumulation vary slowly over lengthscales of order of the thickness, and the bed-undulation amplitude relative to the mean thickness, and the accumulation magnitude relative to a longitudinal velocity magnitude, are bounded by a small parameter ϵ. Series expansions in ϵ for the free-surface profile, velocities and stresses are assumed, and for cold ice the method proposed is to prescribe an initial temperature field and improve it by iteration, rather than to solve the coupled momentum and energy equations. The first-order problem (correction to the parallel-sided slab solution) separates linearly into accumulation and bed-undulation induced effects, and a variety of steady-state illustrations are presented. In particular, the transfer of a harmonic bed undulation to the surface is examined in detail, and extensive numerical results demonstrate the deficiencies of earlier analyses and conclusions. The chapter concludes with a discussion of time-dependent surface perturbations induced by a changing accumulation, and describes a linearized stability analysis for surface waves which suggests, but does not establish, causes of instability.

A complete grounded ice sheet involves a variation of thickness from its maximum to zero at the margins, over a span significantly greater than the maximum thickness, and cannot be described as a perturbation of a parallel-sided slab. The profile is an integral part of the required solution. Chapter 5 presents the shallow-ice approximation in which the longitudinal coordinate is normalized, relative to the maximum thickness, through a small stretching parameter μ so that derivatives with respect to both the thickness and the normalized longitudinal coordinate have equal status. The surface slope relative to the mean bed inclination therefore has magnitude μ, and it is necessary that bedrock undulation slopes do not exceed μ. This global scaling is not appropriate for bed undulation lengths of order of the thickness. The analyses for two different velocity and time scalings are presented. In the first scheme, applicable to a steady inclined glacier with no divide, it is initially supposed that the profile and temperature distribution are known, with μ defined by the aspect ratio, and a perturbation solution is constructed for the velocities and stresses independent of the surface accumulation. An iteration procedure to improve the temperature distribution is proposed. It is then shown how a steady-state surface accumulation condition determines the profile, with a commentary on the limitations of earlier approximate theories. A detailed analysis of the leading and first-order approximations in μ is presented. In the second scheme, applicable to all bed inclinations, the scaled accumulation and normal velocity are immediately taken to have order unity, and the stretching parameter is now determined by the longitudinal momentum balance necessary to describe a bounded sheet with margins, in contrast with an infinite reservoir with horizontal surface. Distinct analyses are required for steep and low-inclination beds, but in both cases, in a steady-state isothermal treatment, the leading-order approximation yields an ordinary differential equation for the profile with sufficient conditions to integrate from a margin. A variety of examples are presented to illustrate the effects of varying the accumulation distribution, the basal sliding law, the viscous relation, and the bed inclination and topography. Subsequent extensions of this approach have determined solutions for prescribed temperature distributions (ignoring energy balance), which confirm the significant influence of temperature, and have yielded a considerably reduced system of partial differential equations as the leading-order approximation for the thermomechanically coupled flow. The analogous scaling for a floating ice shelf yields an integrodifferential equation for the thickness distribution, reducing to a differential equation in the isothermal case, but is not presented in this book.

Chapter 6 is concerned with the response of a glacier to climate, focusing on the effects of a varying surface accumulation with time, and, conversely, asking whether surface accumulation history can be inferred from a history of the advance and retreat of a glacier snout. A kinematic wave equation is derived from the mass balance, which ignores momentum and energy balance but postulates a relation for the total volume flux across a section in terms of the depth and surface slope. The equation is analysed by a linear perturbation about the steady-state solution, and solutions with and without the diffusive term are presented for various flux relations and different accumulation functions. An interesting conclusion is that the history of the glacier snout for only a few years will determine the local current accumulation, while the current local thickness profile depends on the history of accumulation over many hundreds of years. A nonlinear surface-wave equation which incorporates the momentum balance is then derived by applying the shallow-ice approximation with the unsteady surface accumulation condition, and propagation features are analysed by the method of characteristics. A general conclusion from the nonlinear theory is that dispersion is always important, and may possibly dominate diffusion.

The final chapter recognizes the deficiencies of plane-flow theory for describing the three-dimensional flows common in practice. First, channel-flow solutions assuming a plane surface and rectilinear flow are presented for a variety of cross-sections, to describe the effects of valley sides. Next, and new, the shallow-ice approximation with coordinate stretching in both horizontal directions is applied to a grounded sheet with no lateral constraints to obtain the leading-order isothermal equations. Analogous to the plane-flow results, the stress field is given in terms of the (unknown) surface, with the familiar shear stress, depth, surface-gradient relation holding in each horizontal direction. Furthermore, it follows that the horizontal velocity is in the direction of steepest descent of the surface element on the same vertical, and hence does not change direction with depth. In particular, a surface dome (or trough) locates a line of zero horizontal velocity. These are important conclusions which can be tested by observation. The surface accumulation condition now gives a partial differential equation for the surface with unknown margin boundary. Difficulties of the resulting boundary-value problem associated with margin conditions are noted, but this reduced model is a useful starting point for future numerical treatment. The chapter concludes with a description of variational principles which may be a basis for numerical methods.

In summary, the book provides a very detailed mathematical treatment of natural ice flows, and will stand as a reference text for both the practitioner seeking valid models for correlation and prediction, and the theoretician embarking on further developments. The author has taken care to expose the assumptions and physical implications which underlie the theories presented, and to note the many problems not yet treated, formulated properly, or understood. Furthermore, the clarity of his non-native English presentation more than fulfils his prefaced hopes. Detailed algebra and calculation at times detract from the thread, but, echoing the author's introductory warning, this is a book designed not for easy digestion, but rather for the presentation of a rational, unified approach to the subject. It is therefore a work text, to complement and reinforce the easier overviews provided by the various less mathematical glaciology books, and fully meets the aims outlined in the Introduction.

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